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Methodus generalis summandi progressiones

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METHODVS GENERALIS SVMMANDI PROGRESSIONES.

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§. 1.

Propoſui anno praeterito methodum innumeras progreſſiones ſummandi, quae non ſolum ſe ad ſeries algebraicam ſummam habentes extendit, ſed earum etiam, quae algebraice ſummari nequeunt, ſummas a quadraturis curuarum pendentes exhibet. Synthetica tum vſus ſum methodo; generalibus enim aſſumtis formulis quaeſiui ſeries, quarum ſummae iis formulis exprimerentur. Hocque modo plurimas ſeries generales adeptus ſum, quarum ſummas poteram aſſignare. Propoſita igitur quapiam progreſſione ſummanda, neceſſe erat eam cum illis formulis comparare, et indagare, num in aliqua earum contineatur. Potuiſſem autem numerum earum generalium ſerierum in infinitum multiplicare, et propterea ſaepius mihi ſeries occurrerunt, quae etiamſi in datis generalibus non comprehenderentur, ipſa tamen methodo poterant ſummari. Quo igitur facilius magisque in promtu ſit ſeriei cuiuscunque propoſitae ſummam, ſi quidem fieri poteſt, inuenire, communicabo hic methodum analyticam, qua ex ipſius ſeriei natura terminum ſummatorium exuere licet. Latiſſime ea patet; non ſolum enim omnium earum ſerierum, quarum ſummae tot diuerſis modis iam ſunt erutae, ſed infini-

tarum

tarum aliarum ſummas ſimili et facili operatione inuenire docet.

§. 2. Si aeque eſſet facile dato termino generali inuenire ſummatorium, ac inuerſe ex ſummatorio generalem maximum hoc eſſet ſubſidium in ſummatione ſerierum. Poteſt quidem inter terminum ſummatorium et generalem dari aequatio, at quia ex infinitis conſtat terminis, ex ea non multum adiuuamur. At tamen inſigne inde naſcitur compendium, ad progreſſionum algebraicarum ſummas exhibendas. Sit terminus generalis ſeu is, cuius exponens eſt n in progreſſione quacunque t, et terminus ſummatorius ſeu ſumma omnium terminorum a primo vsque ad $t = s$; erit $t = \frac{ds}{1 dn} - \frac{dds}{1.2.dn^2} + \frac{d^3s}{1.2.3.dn^3} - \frac{d^4s}{1.2.3.4.dn^4} +$ etc. in qua aequatione poſitum eſt dn conſtans. Transmutari autem haec aequatio poteſt in hanc $s = \int t dn + \alpha t + \frac{\beta dt}{dn} + \frac{\gamma d^2t}{dn^2} + \frac{\delta d^3t}{dn^3}$ + etc. in qua coefficientes α, β, γ etc. ſequentes habent valores, $\alpha = \frac{1}{2}$; $\beta = \frac{\alpha}{2} - \frac{1}{6}$; $\gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24}$; $\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120}$; $\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720}$; etc. Fiet autem $s = \int t dn + \frac{t}{2} + \frac{dt}{12 dn} - \frac{d^3t}{720 dn^3} + \frac{d^5t}{30240 dn^5}$ etc. Quoties igitur t eiusmodi habet valorem, vt ſeries s praebens vel alicubi abrumpatur, vel fiat ſummabilis, tum ope huius aequationis reperietur s ex t. Euenit autem illud, ſi t eſt functio algebraica rationalis ipſius n, et praeterea ſi eſt fractio, modo n non in determinatorem ingrediatur. E.g. ſit $t = n^2 + 2n$, erit $dt = 2ndn + 2dn$, $ddt = 2dn^2$, $d^3t = 0$ etc. Erit ergo $s = \int (n^2 + 2n) dn + \frac{n^2 + 2n}{2} + \frac{2n + 2}{12} = \frac{n^3}{3} + \frac{3n^2}{2} + \frac{7n}{6} = \frac{2n^3 + 9n^2 + 7n}{6}$.

§. 3. Methodus autem, quam hic ſum expoſiturus, ita ſe habet, vt progreſſio propoſita certis quibusdam operationibus vel ad aliam ſimpliciorem, quae ſummari poteſt, vel iterum ad ſe ipſam reducatur; vtroque enim modo ſumma progreſſionis propoſitae conſtabit. Operationes, quibus in hisce transformationibus vtor, ſunt vel vulgares vt additio, ſubtractio etc. vel ex altiori analyſi, ſumtae vt differentiatio et integratio. Illa quidem aliis ſeriebus non inſeruiunt, niſi quarum ſummatio iam eſt cognita et algebraice aſſignari poteſt; His vero etiam progreſſionum ſummas algebraicas non habentium ſummae a curuarum quadraturis pendentes reperiuntur. Omnes autem ſeries ad quas haec methodus accommodari poteſt, in ſe complectuntur progreſſionem geometricam, et huiusmodi habent formam $\alpha x^a + \beta x^{a+b} + \gamma x^{a+2b} + \delta x^{a+3b} +$ etc. Id quod non impedit, quo minus progreſſio quaecunque in hac forma contineatur.

§. 4. Vt a ſimpliciſſimis incipiam, ſit progreſſio propoſita geometrica, $x^a + x^{a+b} + x^{a+2b} + x^{a+3b} + \text{------} x^{a+(n-1)b}$, in qua extremus terminus eſt is cuius index eſt n, atque hoc in ſequentibus ſemper notetur, terminum vltimum eſſe eum, cuius index eſt n, ne opus habeam indices adſcribere; et proinde etiam ſemper ſummam vsque ad terminum indicis n exhibebo. Ponatur ſumma progreſſionis propoſitae s, erit $s = x^a + x^{a+b} + x^{a+2b} \text{----} + x^{a+(n-1)b}$ tunc fiet $s - x^a = x^{a+b} + x^{a+2b} \text{-----} + x^{a+(n-1)b}$, addatur vtrinque x^{a+nb} et diuidatur per x^b, prodibit $\frac{s - x^a + x^{a+nb}}{x^b}$

$= x^a$

$= x^a + x^{a+b} - - - - - - x^{a+(n-1)b} = s$. Habemus igitur aequationem $s - x^a + x^{a+nb} = s x^b$, ex qua inuenitur $s = \frac{x^a - x^{a+nb}}{1 - x^b}$; quae est summa progressionis geometricae propositae. Est ergo hoc exemplum, quo progressio proposita in se ipsam transmutatur. Si fuerit x fractio vnitate minor et n numerus infinite magnus, erit $x^{a+nb} = 0$ atque $s = \frac{x^a}{1 - x^b}$, summam praebebit progressionis geometricae $x^a + x^{a+b} + x^{a+2b} - - - -$ etc. in infinitum continuatae. Si fuerit $x = 1$ patet esse $s = n$, id vero difficilius apparet ex aequatione $s = \frac{x^a - x^{a+nb}}{1 - x^b}$, quia numerator et denominator euanescunt. Vt vero valor hoc in casu inueniatur, ponatur $x = 1 - \omega$, denotante ω quantitatem infinite paruam, erit $x^a = 1 - a\omega$, $x^{a+nb} = 1 - (a + nb)\omega$ et $x^b = 1 - b\omega$. Hincque fit $s = \frac{nb\omega}{b\omega} = n$. Apparet etiam si terminus generalis seriei fuerit $\alpha x^{a+\overline{n-1}.b}$ fore terminum summatorium $\frac{\alpha x^a - \alpha x^{a+nb}}{1 - x^b}$.

§. 5. Sit nunc proposita ista progressio $x^a + 2 x^{a+b} + 3 x^{a+2b} + - - - - - - + n x^{a+(n-1)b}$, cuius summa ponatur s. Erit $s - x^a = 2 x^{a+b} + 3 x^{a+2b} - - - n x^{a+(n-1)b}$ addatur sequens terminus $(n+1) x^{a+nb}$ et diuidatur per x^b, erit $\frac{s - x^a + (n+1) x^{a+nb}}{x^b} = 2 x^a + 3 x^{a+b} - - - - (n+1) x^{a+(n-1)b}$. Subtrahatur ab hac serie prior scilicet

cet ipsa proposita prodibit $\frac{s - x^a + (n+1)x^{a+nb}}{x^b} - s =$ $x^a + x^{a+b} + x^{a+2b} \ldots\ldots x^{a+(n-1)b} = \frac{x^a - x^{a+nb}}{1 - x^b}$. Ex hac inuenitur $s = \frac{x^a - (n+1)x^{a+nb}}{1-x^b} + \frac{x^{a+b} - x^{a+(n+1)b}}{(1-x^b)^2}$ $= \frac{x^a - (n+1)x^{a+nb} + nx^{a+(n+1)b}}{(1-x^b)^2} = \frac{x^a - x^{a+nb}}{(1-x^b)^2} - \frac{nx^{a+nb}}{1-x^b}$ Qui est terminus summatorius respondens termino generali $nx^{a+(n-1)b}$. Si fuerit $x < 1$ et ponatur $n = \infty$ prodibit seriei propositae in infinitum continuatae summa $=$ $\frac{x^a}{(1-x^b)^2}$. Si autem fiat $x = 1$ prodire debet summa progressionis $1 + 2 + 3 + 4 \ldots\ldots n$, hic vero eadem, quae ante oritur difficultas, numeratore et denominatore euanescentibus; pono igitur iterum $x = 1 - \omega$ erit $1 - x^b = b\omega$; $x^a = 1 - a\omega + \frac{a(a-1)\omega^2}{2}$; $x^{a+nb} = 1 - (a+nb)$ $\omega + \frac{(a+nb)(a+nb-1)\omega^2}{2}$ et $x^{a+(n+1)b} = 1 - (a + (n+1)b)$ $\omega + \frac{(a+(n+1)b)(a+(n+1)b-1)\omega^2}{2}$ fitque $s = \frac{(n^2b^2 + n^2)\omega^2}{2b^2\omega^2} =$ $\frac{nn+n}{2}$. Praeterea si terminus generalis sit $\mathfrak{C}nx^{a+(n-1)b}$ erit terminus summatorius $= \frac{\mathfrak{C}x^a - \mathfrak{C}x^{a+nb}}{(1-x^b)^2} - \frac{\mathfrak{C}nx^{a+nb}}{1-x^b}$.

§. 6. Simili modo inuenientur termini summatorii, si termini generales sint $n^2x^{a+(n-1)b}$, $n^3x^{a+(n-1)b}$ etc. semper enim summatio reducitur ad summationem seriei gradus inferioris. Ex quo intelligitur hac ratione inueniri posse generaliter terminum summatorium spon-

ſpondentem termino generali $(\alpha + \beta n + \gamma n^2 + \text{etc.})$ $x^{a+(n-1)b}$. In his autem abſoluendis longius non immoror, quia iam dudum ſatis ſunt cognita. Ideo haec tantum attuli, vt methodi vis etiam per vulgares operationes pateſcat. Progredior igitur vltra, et quaenam ſeries ope differentiationis et integrationis in ſummam redigi queant, inueſtigabo. Primo quidem etiam progreſſiones algebraicae modo tractatae ſummantur, et ſummae inueniuntur a iam datis non differentes; attamen earum inuentio per has operationes videtur facilior et breuior. Hanc ob rem ab his iterum incipio.

§. 7. Sit progreſſio ſummanda $x + 2x^2 + 3x^3 + 4x^4 - - - nx^n$ ponatur ea $= s$; diuidatur per x et multiplicetur per dx, erit $\frac{sdx}{x} = dx + 2xdx + 3x^2dx - - - - nx^{n-1}dx$, ſumtisque integralibus habetur $\int\frac{sdx}{x} = x + x^2 + x^3 - - - - x^n = \frac{x - x^{n+1}}{1-x}$. Ex aequatione igitur $\int\frac{sdx}{x} = \frac{x - x^{n+1}}{1-x}$ differentiata inuenietur s. Erit enim $\frac{sdx}{x} = \frac{dx - (n+1)x^n dx + nx^{n+1}dx}{(1-x)^2}$, vnde prodit $s = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$, vt ante §. 5. ſi ibi loco a et b ſcribatur 1. Ex hoc intelligi poteſt quomodo progreſſionis $ax^\alpha + (a+b)x^{\alpha+\beta} + (a+2b)x^{\alpha+2\beta} + - - - (a+(n-1)b)x^{\alpha+(n-1)\beta}$ ſumma ſit inuenienda. Ponatur enim haec ſumma quaeſita s, et multiplicetur per $x^\pi dy$, erit $x^\pi s dy = ax^{\alpha+\pi}dy + (a+b)x^{\alpha+\beta+\pi}$

$x^{\alpha+\beta+\pi}dy$ - - - - $(a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dy$. Fiat iam $x^{\alpha+\pi}=y^{a-1}$, et $x^{\alpha+\beta+\pi}=y^{a+b-1}$; erit $x^{\beta}=y^{b}$ et $x=y^{b:\beta}$. Hincque fiet $x^{\alpha+\pi}=y^{(\alpha+\pi)b:\beta}=y^{a-1}$. Ergo erit $\pi=\frac{\beta a-\alpha b-\beta}{b}$. Atque $x^{\alpha+(n-1)\beta+\pi}=y^{a+(n-1)b-1}$ His positis erit $x^{\frac{\beta a-\alpha b-\beta}{b}}s\,dy=ay^{a-1}dy+(a+b)y^{a+b-1}dy+$ - - - $(a+(n-1)b)y^{a+(n-1)b-1}dy$, sumtisque integralibus $\int x^{\frac{\beta a-\alpha b-\beta}{b}}s\,dy=y^{a}+y^{a+b}+$ - - - - $+y^{a+(n-1)b}=\frac{y^{a}-y^{a+nb}}{1-y^{b}}$. Quia vero est $y^{b}=x^{\beta}$; erit $y=x^{\frac{\beta}{b}}$ et $dy=\frac{\beta}{b}x^{\frac{\beta-b}{b}}dx$, hisque substitutis $\frac{\beta}{b}\int x^{\frac{\beta a-\alpha b-b}{b}}$ $s\,dx=\frac{x^{\frac{a\beta}{b}}-x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}$. Haec eadem aequatio potest facilius sine permutatione variabilis x inueniri hoc modo: Multiplicetur progressio proposita per $px^{\pi}dx$, erit $px^{\pi}s\,dx=pax^{\alpha+\pi}dx+$ - - - - - - $p(a+(n-1)b)$ $x^{\alpha+(n-1)\beta+\pi}dx$. Determinentur p et π ita vt sit $\alpha+(n-1)\beta+\pi=p(a+(n-1)b)-1$ seu $\alpha+\pi+(n-1)\beta=ap+(n-1)bp-1$. Ex qua, quia p et π ab n pendere nequeunt, duae resurgunt aequationes $\beta=bp$ et $\alpha+\pi=ap-1$, vnde prodit $p=\frac{\beta}{b}$ et $\pi=\frac{a\beta-\alpha b-b}{b}$. His substitutis, et integralibus sumtis, proueniet vt ante $\frac{\beta}{b}\int x^{\frac{a\beta-\alpha b-b}{b}}s\,dx=x^{\frac{a\beta}{b}}+x^{\frac{a\beta+b\beta}{b}}$ - - $+x^{\frac{a\beta+(n-1)b\beta}{b}}=\frac{x^{\frac{a\beta}{b}}-x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}$.

§. 8. Sit progressionis propositae terminus ordine n, hic $(an+b)(cn+e)x^{\alpha+(n-1)\beta}$; ponatur huius terminus

minus ſummatorius s: erit $s = (a+b)(c+e)x^{\alpha} + (2a+b)(2c+e)x^{\alpha+\beta} + - - - - - - + (an+b)(cn+e)x^{\alpha+(n-1)\beta}$; multiplicetur per $px^{\pi}dx$, fiet $psx^{\pi}dx = p(a+b)(c+e)x^{\alpha+\pi}dx + - - - - - - + p(an+b)(cn+e)x^{\alpha+(n-1)\beta+\pi}dx$. Sit $pcn + pe = \alpha + n\beta - \beta + \pi + 1$, debebit eſſe $p = \frac{\beta}{c}$ et $\pi = \frac{\beta e + \beta c - \alpha c - c}{c}$. Ergo ſumtis integralibus erit $\frac{\beta}{c}\int \pi^{\pi} s dx = (a+b)x^{\alpha+\pi+1} + - - - - (an+b)x^{\alpha+(n-1)\beta+\pi+1}$. Multiplicetur denuo per $qx^{\rho}dx$, erit $\frac{\beta}{c}qx^{\varrho}dx\int x^{\pi}sdx = q(a+b)x^{\alpha+\pi+\rho+1}dx + - - - - - q(an+b)x^{\alpha+(n-1)\beta+\pi+\varrho+1}dx$, fiatque $anq + bq = \alpha + n\beta - \beta + \pi + \varrho + 2$, hinc erit $q = \frac{\beta}{a}$ et $\varrho = \frac{\beta b - \alpha a + \beta a - \pi a - 2a}{a} = \frac{\beta bc - ac - \beta ae}{ac}$. Sumtisque integralibus proueniet $\frac{\beta^2}{ac}\int x^{\rho}dx\int x^{\pi}sdx = x^{\alpha+\pi+\rho+2} + - - - - + x^{\alpha+(n-1)\beta+\pi+\varrho+2} = \frac{x^{\alpha+\pi+\rho+2} - x^{\alpha+n\beta+\pi+\rho+2}}{1-x^{\beta}}$

ſeu haec aequatio $\frac{\beta^2}{ac}\int x^{\frac{\beta bc - \beta z e - ac}{ac}}dx\int x^{\frac{\beta e + \beta c - \alpha c - c}{c}}sdx = \frac{x^{\frac{\beta(c+b)}{a}} - x^{\frac{\beta(a+b+na)}{a}}}{1-x^{\beta}} = x^{\frac{\beta(a+b)}{a}}\left(\frac{1-x^{n\beta}}{1-x^{\beta}}\right)$. Simili modo operatio eſt inſtituenda, ſi plures duobus factores fuerint in termino generali, ex quo ſimul apparet, tot prodire ſigna integralia, quot ſunt factores in coefficiente termini generalis.

§. 9. Si fuerit progreſſionis ſummandae terminus generalis $\frac{x^{\alpha+(n-1)\beta}}{an+b}$, operatio a priori in hoc tantum differt, quod hic differentiatione abſolui debeat, quod

 ibi

ibi integralibus ſumendis perficiebatur. Sit igitur terminus ſummatorius quaeſitus s, erit $s = \frac{x}{a+b} + \cdots + \frac{x^{\alpha+(n-1)\beta}}{an+b}$, atque $p x^{\pi} s = \frac{p x^{\alpha+\pi}}{a+b} + \cdots + \frac{p x^{\alpha+(n-1)\beta+\pi}}{an+b}$. Sumantur differentialia prodibit $p x^{\pi} ds + p \pi x^{\pi-1} s dx = \frac{p(\alpha+\pi) x^{\alpha+\pi-1} dx}{a+b} + \cdots + \frac{p(\alpha+n\beta-\beta+\pi) x^{\alpha+(n-1)\beta+\pi-1} dx}{an+\beta}$. Fiat $p\alpha + pn\beta - p\beta + p\pi = an+b$, erit $p = \frac{a}{\beta}$ et $\pi = \beta - \alpha + \frac{b\beta}{a}$. Ergo $\frac{a x^{\beta-\alpha+\frac{b\beta}{a}} ds + (a\beta - a\alpha + b\beta) x^{\beta-\alpha+\frac{b\beta}{a}-1} s dx}{\beta dx} = x^{\frac{a\beta+b\beta-a}{a}} + \cdots + x^{\frac{na\beta+b\beta-a}{a}} = x^{\frac{a\beta+b\beta-a}{a}} \left(\frac{1-x^{n\beta}}{1-x^{\beta}}\right)$. Seu $\frac{a}{\beta} x^{\frac{a\beta-a\alpha+b\beta}{a}} s = \int x^{\frac{a\beta+b\beta-a}{a}} dx \left(\frac{1-x^{n\beta}}{1-x^{\beta}}\right)$ vel $s = \frac{\beta}{a} x^{\frac{a\alpha-a\beta-b\beta}{a}} \int x^{\frac{a\beta+b\beta-a}{a}} dx \left(\frac{1-x^{n\beta}}{1-x^{\beta}}\right)$. In hac formula integrale ita debet accipi vt poſito $x=0$, ipſum euaneſcat. Si deſideretur ſumma ſeriei propoſitae in infinitum continuatae, fiet $n=\infty$ et $s = \frac{\beta}{a} x^{\frac{a\alpha-a\beta-\beta b}{a}} \int \frac{x^{\frac{a\beta+b\beta-a}{a}} dx}{1-x^{\beta}}$. Si ſit $x=1$. in expreſſione quidem ſumma s, quia differentialia inſunt, non poteſt poni $x=1$, ſed poſt integrationem fiat $x=1$. Attamen perinde

eſt

est, quales numeri loco α et β substituantur, sit igitur $\alpha = \beta = 1$. Erit $s = \frac{1}{a+b} + \frac{1}{2a+b} - - - + \frac{1}{na+b} = \frac{1}{a x^{\frac{a}{b}}} \int x^{\frac{b}{a}} dx \left(\frac{1-x^n}{1-x}\right)$. Atque post integrationem fieri debet $x = 1$. Quemadmodum in dissertatione de summationibus initio citata inueneram.

§. 10. Sit proposita progressio, cuius terminus ordine n est $\frac{x^n}{(an+b)(cn+e)}$, assumo hic tantum x^n loco $x^{\alpha+(n-1)\beta}$ tum compendii ergo, tum quia haec potentia in illam facili negotio potest transmutari. Sit terminus summatorius s, erit $p x^\pi s = \frac{p x^{\pi+1}}{(a+b)(c+e)} + - - - + \frac{p x^{\pi+n}}{(an+b)(cn+e)}$. Adeoque $\frac{\text{diff.}\, p x^\pi s}{dx} = \frac{p(\pi+1)x^\pi}{(a+b)(c+e)} + - - - - - \frac{p(\pi+n)x^{\pi+n-1}}{(an+b)(cn+e)}$. Fiat $p\pi + pn = an + b$, erit $p = a$ et $\pi = \frac{b}{a}$. Ergo habetur $\frac{a d(x^{\frac{b}{a}} s)}{dx} = \frac{x^{\frac{b}{a}}}{c+e} + - - - - - + \frac{x^{\frac{b}{a}+n-1}}{cn+e}$. Multiplicetur denuo per $p x^\pi$, erit $\frac{a p x^\pi d(x^{\frac{b}{a}} s)}{dx} = \frac{p x^{\frac{b}{a}+\pi}}{c+e} + - - - - - + \frac{p x^{\frac{b}{a}+n+\pi-1}}{cn+e}$. Hincque prodit $\frac{apd(x^\pi d(x^{\frac{b}{a}} s))}{dx} = \frac{p(\frac{b}{a}+\pi)x^{\frac{b}{a}+\pi-1}}{c+e} + - - + \frac{p(\frac{b}{a}+n+\pi-1)x^{\frac{b}{a}+n+\pi-2}}{cn+e}$

Fiat

Fiat $\frac{pb}{a}+pn+p\pi-p=cn+e$; erit $p=c$ et $\pi=1-\frac{b}{a}+\frac{e}{c}$. His ſubſtitutis emerget iſta aequatio $\frac{acd(x^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s))}{dx^2}=x^{\frac{e}{c}}+\ldots+x^{\frac{e}{c}+n-1}=x^{\frac{e}{c}}\left(\frac{1-x^n}{1-x}\right)$

Sumantur iterum integralia, erit $\frac{acx^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s)}{dx}=$ $\int x^{\frac{e}{c}}dx\left(\frac{1-x^n}{1-x}\right)$: hincque $s=\frac{1}{acx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}}dx$ $\left(\frac{1-x^n}{1-x}\right)=\frac{x^{\frac{b}{a}-\frac{e}{c}}\int x^{\frac{e}{c}}dx(\frac{1-x^n}{1-x})-\int x^{\frac{b}{a}}dx(\frac{1-x^n}{1-x})}{(bc-ae)x^{\frac{b}{a}}}$. Caſus hic notandus eſt, ſi $bc=ae$, quo fit $s=\frac{0}{0}$. Erit autem iuxta priorem formam $s=\frac{1}{acx^{\frac{b}{a}}}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\left(\frac{1-x^n}{1-x}\right)$ quae mutatur in hanc $s=\frac{lx\int x^{\frac{b}{a}}dx(\frac{1-x^n}{1-x})-\int x^{\frac{b}{a}}dx(\frac{1-x^n}{1-x})lx}{acx^{\frac{b}{a}}}$

Caſus hic accidit, ſi denominatores $(an+b)(cn+e)$ fuerint quadrata vel horum quaedam multipla. Si fuerit $x=1$, haec ſubſtitutio vt ante demum poſt integrationem fieri debet in quantitatibus ſigna integralia prae ſe habentibus, at in finitis ſtatim fieri poteſt $x=1$.

Erit ergo $s=\frac{\int(x^{\frac{e}{c}}-x^{\frac{b}{a}})dx(\frac{1-x^n}{1-x})}{bc-ae}$. Ex quo apparet ſi $x^{\frac{e}{c}}-x^{\frac{b}{a}}$ poteſt diuidi per $1-x$ ſummam progreſſionis eſſe algebraicam. At caſu quo $bc=ae$, fiet $lx=0$,

$= o$, ſi ſcilicet ſit $x = 1$. Quocirca erit $s = \frac{\int x^{\frac{b}{a}} dx \left(\frac{1-x^n}{1-x}\right) lx}{m\,dc}$.

§. 11. Simili modo intelligitur ſi n in denominatore 3 pluresue dimenſiones habeat, quomodo ſummam inueniri oporteat, ita vt opus non ſit pluribus exemplis operationem illuſtrare. Sit progreſſio propoſita haec cuius terminus generalis eſt $\frac{x^n}{(an+b)(cn+e)(fn+g)}$ ſumma huius ſit s. Haec progreſſio eodem, quo praecedente §. modo tractata dabit poſt duas differentiones

$$\frac{acd\left(x^{1-\frac{b}{a}+\frac{e}{c}} d\left(x^{\frac{b}{a}} s\right)\right)}{dx^2} = \frac{x^{\frac{e}{c}}}{f+g} + \text{- - - - - - -} \frac{x^{\frac{e}{c}+n-1}}{nf+g}$$

$= (\text{p. §. 9.})\ \frac{1}{f} x^{\frac{e}{c}-\frac{g}{f}-1} \int x^{\frac{f}{g}} dx \left(\frac{1-x^n}{1-x}\right)$ ſumantur integralia erit $\frac{acfx^{1-\frac{a}{b}+\frac{e}{c}} d\left(x^{\frac{b}{a}} s\right)}{dx} = \int x^{\frac{e}{c}-\frac{g}{f}-1} dx \int x^{\frac{g}{f}} dx \left(\frac{1-x^n}{1-x}\right)$, et denuo $acfx^{\frac{b}{a}} s = \int x^{\frac{b}{a}-\frac{e}{c}-1} dx \int x^{\frac{e}{c}-\frac{g}{f}-1} dx \int x^{\frac{g}{f}} dx \left(\frac{1-x^n}{1-x}\right)$, adeoque $s = \frac{1}{acfx^{\frac{b}{a}}} \int x^{\frac{b}{a}-\frac{e}{c}-1} dx \int x^{\frac{e}{c}-\frac{g}{f}-1} dx \int x^{\frac{g}{f}} dx \left(\frac{1-x^n}{1-x}\right)$. Ne plura ſigna integralia poſt ſe inuicem ſint poſita, haec forma in ſequentem transmutari poteſt $s = \frac{fx^{-\frac{g}{f}} \int x^{\frac{g}{f}} dx \left(\frac{1-x^n}{1-x}\right)}{(bf-ag)(ef-cg)} + \frac{cx^{-\frac{e}{c}} \int x^{\frac{e}{c}} dx \left(\frac{1-x^n}{1-x}\right)}{(bc-ae)(cg-ef)}$

+

$+\frac{ax^{\frac{-b}{a}}\int x^{\frac{b}{a}}dx\left(\frac{1-x^n}{1-x}\right)}{(ae-bc)(ag-bf)}$. Ex hoc ſimul apparet, ſi plures fuerint factores in termino generali, quam formam habitura ſit ſumma. Sit enim terminus generalis $\frac{x^n}{(an+b)(cn+e)(fn+g)(hn+k)}$ erit terminus ſummatorius $s=\frac{1}{acfhx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}-\frac{k}{h}-1}dx\int x^{\frac{k}{h}}dx\left(\frac{1-x^n}{1-x}\right)=\frac{ax^{\frac{-b}{a}}\int x^{\frac{b}{a}}dx\left(\frac{1-x^n}{1-x}\right)}{(ae-bc)(ag-bf)(ak-bh)}+\frac{cx^{\frac{-e}{c}}\int x^{\frac{e}{c}}dx\left(\frac{1-x^n}{1-x}\right)}{(bc-ae)(cg-ef)(ck-eh)}+\frac{fx^{\frac{-g}{f}}\int x^{\frac{g}{f}}dx\left(\frac{1-x^n}{1-x}\right)}{(bf-ag)(ef-cg)(fk-gh)}+\frac{kx^{\frac{-k}{h}}\int x^{\frac{k}{h}}dx\left(\frac{1-x^n}{1-x}\right)}{(bh-ak)(eh-ck)(gh-fk)}$. Si deſideretur ſumma caſu, quo $x=1$. erit pro termino generali $\frac{1}{(an+b)(cn+e)(fn+g)}$ terminus ſummatorius $s=\frac{\int dx\left(\frac{1-x^n}{1-x}\right)\left((aef-bcf)x^{\frac{g}{f}}+(bcf-acg)x^{\frac{e}{c}}+(acg-aef)x^{\frac{b}{a}}\right)}{(ae-bc)(ag-bf)(cg-ef)}$. Quoties igitur quantitas in $dx\left(\frac{1-x^n}{1-x}\right)$ ducta dividi poteſt per $1-x$ tunc progreſſio propoſita algebraicam habet ſummam. Accidit hoc ſi $\frac{b}{a}-\frac{e}{c}$ et $\frac{e}{c}-\frac{g}{f}$ ſunt numeri integri. Praeterea hoc etiam eſt notandum omnes huiusmodi progreſſiones vel algebraice eſſe ſummabiles, vel a logarithmis ſiue realibus

libus siue imaginariis pendere, neque vllam aliam quadraturam huiusmodi progressione posse exprimi.

§. 12. At cum difficile sit has formulas ad eos casus accommodare, quibus denominatorum factores sunt aequales, libet hic hos casus in specie tractare: sit itaque progressionis summandae terminus generalis $\frac{x^n}{(an+b)^3}$ et summatorius s, erit $s=\frac{1}{a^3 x^{\frac{b}{a}}}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\frac{b}{a}}\,dx\left(\frac{1-x^n}{1-x}\right)$ id quod sequitur ex §. 11. vbi fit $c=f=a$ et $e=g=b$: haec forma transmutata abit in hanc $s=\frac{\frac{1}{2}(lx)^2\int x^{\frac{b}{a}}\,dx\left(\frac{1-x^n}{1-x}\right)-lx\int x^{\frac{b}{a}}\,dx\,lx\left(\frac{1-x^n}{1-x}\right)+\frac{1}{2}\int x^{\frac{b}{a}}\,dx\left(\frac{1-x^n}{1-x}\right)(lx)^2}{a^3 x^{\frac{b}{a}}}$. Si autem fuerit terminus generalis $\frac{x^n}{(an+b)^4}$, erit $s=\frac{(lx)^3\int x^{\frac{b}{a}}\,dx\left(\frac{1-x^n}{1-x}\right)-3(lx)^2\int x^{\frac{b}{a}}\,dx\,lx\left(\frac{1-x^n}{1-x}\right)+3lx\int x^{\frac{b}{a}}\,dx(lx)\left(\frac{1-x^n}{1-x}\right)-\int x^{\frac{b}{a}}\,dx(lx)^3\left(\frac{1-x^n}{1-x}\right)}{6a^4 x^{\frac{b}{a}}}$

Ex his apparet quomodo pro reliquis potentiis valor ipsius s progrediatur: generaliter enim si terminus generalis est $\frac{x^n}{(an+b)^m}$, erit summa $s=\frac{(lx)^{m-1}\int x^{\frac{b}{a}}\,dx\left(\frac{1-x^n}{1-x}\right)-\left(\frac{m-1}{1}\right)(lx)^{m-2}\int x^{\frac{b}{a}}\,dx\,lx\left(\frac{1-x^n}{1-x}\right)+\left(\frac{m-1}{1}\right)\left(\frac{m-2}{2}\right)(lx)^{m-3}\int x^{\frac{b}{a}}\,dx(lx)^2\left(\frac{1-x^n}{1-x}\right)-\text{etc.}}{1.\,2.\,3.\,\text{----}\,(m-1)a^m x^{\frac{b}{a}}}$

—etc. Valores hi multo fiunt fimpliciores, fi ponatur $x=1$, erit enim $lx=o$. Termino generali enim $\frac{1}{(an+b)^2}$ refpondet hic fummatorius $\frac{\int x^{\frac{b}{a}} dx (l\frac{1}{x})(\frac{1-x^n}{1-x})}{1.a^2}$: termino generali $\frac{1}{(an+b)^3}$ hic $\frac{\int x^{\frac{b}{a}} dx (l\frac{1}{x})^2 (\frac{1-x^n}{1-x})}{1.2.a^3}$; atque termino generali $\frac{1}{(an+b)^m}$ hic $\frac{\int x^{\frac{b}{a}} dx (l\frac{1}{x})^{m-1} (\frac{1-x^n}{1-x})}{1.2.3.---(m-1)a^m} = \frac{\int x^{\frac{b}{a}} dx (l\frac{1}{x})^{m-1} (\frac{1-x^n}{1-x})}{a^m \int dx (l\frac{1}{x})^{m-1}}$; quae integralia ita debent accipi vt pofito $x=0$ tota fumma euanefcat, tum autem poni debet $x=1$, et quantitas refultans vera erit fumma. Porro notetur fi fumma defideretur in infinitum continuatae progreffionis, vbique tantum fcribi debere $\frac{1}{1-x}$ loco $\frac{1-x^n}{1-x}$.

§. 13. Duae iam pertractatae funt progreffionum claffes, quarum illa habebat terminum generalem Ax^n haec vero $\frac{x^n}{A}$ denotante A quantitatem algebraicam ex n et conftantibus conftantem, ita tamen, vt n non habeat alios exponentes, nifi integros affirmatiuos. Ex his oritur tertia claffis pro termino generali habens $\frac{Ax^n}{B}$, vbi A et B eiusdem modi quantitates algebraicas defignant. Talis progreffio reducitur etiam ad progreffionem geometricam tollendo numeratorem A ope integra-

tionis

tionis, et denominatorem B ope differentiationis, quemadmodum in vtraque pertractata ſeorſim factum eſt. Sit progreſſionis ſummandae terminus generalis $\frac{(\alpha n+\beta)x^n}{(an+b)}$, huius terminus ſummatorius ponatur s; erit $s = \frac{(\alpha+\beta)x}{a+b} + - - - - + \frac{(\alpha n+\beta)x^n}{an+b}$. Multiplicetur haec aequatio per px^π, erit $px^\pi s = \frac{p(\alpha+\beta)x^{\pi+1}}{a+b} + - - - - + \frac{p(\alpha n+\beta)x^{n+\pi}}{an+b}$ ſumantur differentialia, erit $pd(x^\pi s) = \frac{p(\pi+1)(\alpha+\beta)x^\pi dx}{a+b} + - - - - + \frac{p(n+\pi)(\alpha n+\beta)x^{n+\pi-1}dx}{an+b}$, fiat $pn + p\pi = an + b$, erit $p = a$ et $\pi = \frac{b}{a}$. Ergo eſt $ad(x^{\frac{b}{a}}s) = (\alpha+\beta)x^{\frac{b}{a}}dx + - - - + (\alpha n+\beta)x^{\frac{b+n-1}{a}}dx$. Multiplicetur denuo per px^π erit $apx^\pi d(x^{\frac{b}{a}}s) = p(\alpha+\beta)x^{\frac{b+\pi}{a}}dx + - - - - - + p(\alpha n+\beta)x^{\frac{b+\pi+n-1}{a}}dx$. Sumantur integralia habebitur $ap\int x^\pi d(x^{\frac{b}{a}}s) = \frac{ap(\alpha+\beta)x^{\frac{b+\pi+1}{a}}}{b+a\pi+a} + - - - + \frac{ap(\alpha n+\beta)x^{\frac{b+\pi+n}{a}}}{b+a\pi+an}$. Fiat $a\alpha pn + a\beta p = an + a\pi + b$; erit $p = \frac{1}{\alpha}$ et $\pi = \frac{\beta}{\alpha} - \frac{b}{a}$. Propterea eſt $\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}} d(x^{\frac{b}{a}}s) = x^{\frac{\beta+1}{\alpha}} + - - - + x^{\frac{\beta+n}{\alpha}} = x^{\frac{\beta+1}{\alpha}}\left(\frac{1-x^n}{1-x}\right)$. Ex hac aequatione prodit $s = \frac{\alpha\int x^{\frac{a}{a}-\frac{\beta}{\alpha}} d\left(x^{\frac{\beta+\alpha}{\alpha}}\left(\frac{1-x^n}{1-x}\right)\right)}{ax^{\frac{b}{a}}}$. Si fuerit terminus generalis $\frac{(\alpha n+\beta)(\gamma n+\delta)x^n}{an+b}$, huiusque ſummatorius ponatur s, prodibit

dibit iisdem, quibus modo, abſolutis operationibus, $\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=(\gamma+\delta)x^{\frac{\beta}{\alpha}+1}+----+(\gamma n+\delta)x^{\frac{\beta}{\alpha}+n}$, multiplicetur iterum per $px^{\pi}dx$ et ſumantur integralia, prodibit $\frac{ap}{\alpha}\int x^{\pi}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=\frac{\alpha p(\gamma+\delta)x^{\frac{\beta}{\alpha}+\pi+2}}{\beta+\alpha\pi+2\alpha}+----+\frac{\alpha p(\gamma n+\delta)x^{\frac{\beta}{\alpha}+\pi+n+1}}{\beta+\alpha\pi+\alpha n+\alpha}$. Fiat $\alpha\gamma pn+\alpha\delta p=\alpha+\beta+\alpha\pi+\alpha n$, erit $p=\frac{1}{\gamma}$ et $\pi=\frac{\delta}{\gamma}-\frac{\beta-1}{\alpha}$. Ergo $\frac{a}{\alpha\gamma}\int x^{\frac{\delta}{\gamma}-\frac{\beta-1}{\alpha}}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=x^{\frac{\delta}{\gamma}+1}+---+x^{\frac{\delta}{\gamma}+n}=x^{\frac{\delta}{\gamma}+1}(\frac{1-x^{n}}{1-x})$. Quare $s=\frac{\alpha\gamma\int x^{\frac{b}{a}-\frac{\beta}{\alpha}}d(x^{\frac{\beta}{\alpha}-\frac{\delta}{\gamma}+1}d(x^{\frac{\delta}{\gamma}+1}(\frac{1-}{1-}}{ax^{\frac{b}{a}}dx}$

Sed huiusmodi progreſſionibus ſummandis diutius non immoror, ſufficit enim methodum tradidiſſe, qua omnes ſummari poſſint. Interim tamen et id valet, quod §. 11. dixi, omnes ſcilicet huiusmodi progreſſiones vel algebraice poſſe ſummari, vel ſummam a logarithmis ſiue realibus ſiue imaginariis pendere.

§. 14. Progredior nunc ad aliud progreſſionum genus, quarum termini generales algebraice exprimi non poſſunt, ſed quae ad claſſem ſerierum hypergeometricarum pertinent. Huiusmodi ſeries eſt $(\alpha+\beta)x+(\alpha+\beta)(2\alpha+\beta)x^{2}+---(\alpha+\beta)(2\alpha+\beta)-----(\alpha n+\beta)x^{n}$. Ponatur huius ſumma s, et multiplicetur per px^{π}, erit $px^{\pi}s=p(\alpha+\beta)x^{\pi+1}+----+p(\alpha+\beta)$
$(2\alpha+\beta)$

$(2\alpha+\beta) \cdots (\alpha n+\beta)x^{n+\pi}$. Et huius in dx ductae integralis $p\int x^{\pi} s\,dx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \cdots + \frac{p(\alpha+\beta)(2\alpha+\beta)(2\alpha+\beta) \cdots (\alpha n+\beta)x^{n+\pi+1}}{n+\pi+1}$, fiat $p\alpha n + p\beta = n+\pi+1$ erit $p = \frac{1}{\alpha}$ et $\pi = \frac{\beta}{\alpha} - 1$. Vnde prodit $\frac{1}{\alpha}\int x^{\frac{\beta}{\alpha}-1} s\,dx = x^{\frac{\beta}{\alpha}+1} + (\alpha+\beta)x^{\frac{\beta}{\alpha}+2} + \cdots + (\alpha+\beta)(2\alpha+\beta) \cdots (\alpha(n-1)+\beta)x^{\frac{\beta}{\alpha}+n}$. Diuidatur per $x^{\frac{\beta}{\alpha}+1}$ habebitur $\frac{\int x^{\frac{\beta}{\alpha}-1} s\,dx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = (\alpha+\beta)x + \cdots (\alpha+\beta)(2\alpha+\beta) \cdots (\alpha(n-1)+\beta)x^{n-1}$. Quae est ipsa progressio proposita truncata termino vltimo. Erit igitur $\frac{\int x^{\frac{\beta}{\alpha}-1} s\,dx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = s - (\alpha+\beta)(2\alpha+\beta) \cdots (\alpha n+\beta) = s - A$. Huiusmodi autem formas finita expressione exposui in alia iam praelecta dissertatione de terminis generalibus progressionum transcendentalium, ex qua si libet finitus valor loco A desumi potest. Erit ergo $\int x^{\frac{\beta}{\alpha}-1} s\,dx = \alpha x^{\frac{\beta}{\alpha}+1} + \alpha x^{\frac{\beta}{\alpha}+1} s - \alpha A x^{\frac{\beta}{\alpha}+n+1}$, atque $x^{\frac{\beta}{\alpha}-1} s\,dx = (\alpha+\beta)x^{\frac{\beta}{\alpha}}dx + (\alpha+\beta)x^{\frac{\beta}{\alpha}} s\,dx + \alpha x^{\frac{\beta}{\alpha}+1} ds - (\alpha+\beta+\alpha n)Ax^{\frac{\beta}{\alpha}+n}dx$ seu $s\,dx = (\alpha+\beta)x\,dx + (\alpha+\beta)x s\,dx + \alpha x^2 ds - (\alpha+\beta+\alpha n)Ax^{n+1}dx$. Ex qua aequatione valor ipsius s erutus dabit summam progressionis propositae. Fieri etiam potest, vt factores in termino sequente non vno tantum, sed duobus pluribusue

ribusue augeantur. Accedant ſemper duo de nouo, vt prodeat iſta progreſſio $(\alpha+\beta)x+(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)x^2+\text{-----}+(\alpha+\beta)(2\alpha+\beta)\text{--}(\alpha(2n-1)+\beta)x^n$. Huius ſumma vocetur s, erit $p\int x^\pi s\,dx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2}+\text{-----}+\frac{p(\alpha+\beta)(2\alpha+\beta)\text{---}p(\alpha(2n-1)+\beta)x^{n+\pi+1}}{n+\pi+1}$. Fiat $2pan-p\alpha+p\beta=n+\pi+1$ erit $p=\frac{1}{2\alpha}$, et $\pi=\frac{\beta-3\alpha}{2\alpha}$. Vnde $\frac{\int x^{\frac{\beta-3\alpha}{2\alpha}}s\,dx}{2\alpha} = x^{\frac{\beta+\alpha}{2\alpha}}+\text{-----}+(\alpha+\beta)(2\alpha+\beta)\text{---}(\alpha(2n-2)+\beta)x^{n+\frac{\beta-\alpha}{2\alpha}}$. Atque iterum $\frac{p\int x^\pi dx\int x^{\frac{\beta-3\alpha}{2\alpha}}s\,dx}{2\alpha} = \frac{2\alpha p\,x^{\frac{\beta+3\alpha}{2\alpha}+\pi}}{\beta+3\alpha+2\alpha\pi}+\text{-----}+\frac{2\alpha p(\alpha+\beta)\text{-----}(\alpha(2n-2)+\beta)x^{n+\pi+\frac{\beta+\alpha}{2\alpha}}}{\beta+\alpha+2\alpha n+2\pi\alpha}$. Fiat $4p\alpha^2n-4p\alpha^2+2p\alpha\beta=2\alpha n+2\pi\alpha+\alpha+\beta$; erit $p=\frac{1}{2\alpha}$ et $\pi=\frac{\beta-2\alpha}{2\alpha}-\frac{\alpha-\beta}{2\alpha}=-\frac{3}{2}$; conſequenter $\frac{\int x^{-\frac{3}{2}}dx\int x^{\frac{\beta-3\alpha}{2\alpha}}s\,dx}{4\alpha^2x^{\frac{\beta}{2\alpha}}}-\frac{1}{\beta}=(\alpha+\beta)x+\text{----}(\alpha+\beta)(2\alpha+\beta)\text{---}(\alpha(2n-3)+\beta)x^{n-1}=s-Ax^n$ poſito $A=(\alpha+\beta)(2\alpha+\beta)\text{----}(\alpha(2n-1)+\beta)$. Ex qua aequatione s innoteſcit.

§. 15. Simili modo operationem inſtitui oportet, ſi in coefficiente termini ſequentis, tres pluresue factores de nouo accedant. De quo notandum eſt,

est, tot semper in aequatione resultante signa integralia sibi inuicem esse iuncta quot sunt factores, quibus sequens quisque terminus augetur. Ita progressionis $(\alpha+\beta)$ [illegible] $+ (\alpha+\beta) ---- (\alpha(3n-2)+\beta)x^n$ summa s determinabitur ex hac aequatione $\frac{\int x^{-\frac{4}{3}} dx \int x^{-\frac{4}{3}} dx \int x^{\frac{\beta-5\alpha}{3\alpha}} s\,dx}{27\alpha^3 x^{\frac{\beta-\alpha}{3\alpha}}}$ [illegible] $(\alpha+\beta) ---- (\alpha(3n-2)+\beta)x^n$. Ex qua, vt inductio ad sequentes casus fieri possit, notandum est, [illegible] esse terminum progressionis propositae ante primum seu eum cuius index est o. Si factores qui in potentias ipsius x ducuntur non constituant progressionem arithmeticam, sed aliam algebraicam altioris ordinis, operatio similiter debet institui; vt sit progressio proposita $(\alpha+\beta)(\gamma+\delta)x + ---- + (\alpha+\beta)(2\alpha+\beta) ---- (\alpha n+\beta)(\gamma+\delta)(2\gamma+\delta) --- (\gamma n+\delta)x^n$ ponatur huius summa s, erit $p\int x^\pi s\,dx = \frac{p(\alpha+\beta)(\gamma+\delta)x^{\pi+2}}{\pi+2} + ---- + \frac{p(\alpha+\beta) ---- (\alpha n+\beta)(\gamma+\delta) ---- (\gamma n+\delta)x^{n+\pi+1}}{n+\pi+1}$.

Ponatur $p\gamma n + p\delta = n+\pi+1$; erit $p = \frac{1}{\gamma}$; et $\pi = \frac{\delta-\gamma}{\gamma}$. Ergo $\frac{\int x^{\frac{\delta-\gamma}{\gamma}} s\,dx}{\gamma} = (\alpha+\beta)x^{\frac{\delta+\gamma}{\gamma}} + ---- (\alpha+\beta) ---- (\alpha n+\beta)(\gamma+\delta) ---- (\gamma(n-1)+\delta)x^{\frac{n+\delta}{\gamma}}$.

Porro erit $\frac{p\int x^\pi dx \int x^{\frac{\delta-\gamma}{\gamma}} s\,dx}{\gamma} = \frac{\gamma p(\alpha+\beta)x^{\frac{\delta+2\gamma}{\gamma}+\pi}}{\delta+2\gamma+\pi\gamma} + ---- + \frac{\gamma p(\alpha+\beta) ---- (\alpha n+\beta)(\gamma+\delta) ---- (\gamma(n-1)+\delta)x^{n+\pi+\frac{\delta+\gamma}{\gamma}}}{\gamma n+\pi\gamma+\delta+\gamma}$.

Fiat $p\alpha\gamma n + p\beta\gamma = \gamma n + \pi\gamma + \delta + \gamma$, erit $p = \frac{1}{\alpha}$, et

$\frac{1}{\alpha}$, et $p = \frac{\beta}{\alpha} - \frac{\delta}{\gamma} - 1 = \frac{\beta\gamma - \alpha\delta - \alpha\gamma}{\alpha\gamma}$. Ergo

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}} dx \int x^{\frac{\delta-\gamma}{\gamma}} s\,dx}{\alpha\gamma} = x^{\frac{\beta+\alpha}{\alpha}} + \text{-----} + (\alpha+\beta)$$

$$\text{----}\,(\alpha(n-1)+\beta)(\gamma+\delta)\text{---}(\gamma(n-1)+\delta)x^{\frac{\beta}{\alpha}+n}$$

Confequenter $\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}} dx \int x^{\frac{\delta-\gamma}{\gamma}} s\,dx}{\alpha\gamma x^{\frac{\beta+\alpha}{\alpha}}} - 1 = s - ABx^n$.

Pofito $A = (\alpha+\beta)\text{--}(\alpha n+\beta)$ et $B = (\gamma+\delta)\text{---}(\gamma n+\delta)$. Hic eft cafus fi progreffionis, quam factores conficiunt terminus generalis eft $(\alpha n+\beta)(\gamma n+\delta)$ feu $\alpha\gamma n^2 + (\alpha\delta+\beta\gamma)n + \beta\delta$. Comprehenduntur ergo fub hac forma omnes progreffiones ordinis fecundi. Superior autem formula ex qua s determinatur transmutatur in hanc $\frac{\int x^{\frac{\delta-\gamma}{\gamma}} s\,dx}{(\beta\gamma-\alpha\delta)x^{\frac{\delta+\gamma}{\gamma}}} + \frac{\int x^{\frac{\beta-\alpha}{\alpha}} s\,dx}{(\alpha\delta-\beta\gamma)x^{\frac{\beta+\alpha}{\alpha}}} = 1 + s - ABx^n$. Ex qua facilius forma fequentium intelligi poteft.

§. 16. Confiderabo nunc harum ferierum reciprocas, in quibus potentiae ipfius x funt diuifae per id, per quod ante erant multiplicatae. Sit igitur feries fummanda haec $\frac{x}{\alpha+\beta} + \frac{x^2}{(\alpha+\beta)(2\alpha+\beta)} + \frac{x^3}{(\alpha+\beta)\text{---}(3\alpha+\beta)} + \text{------} + \frac{x^n}{(\alpha+\beta)\text{-----}(\alpha n+\beta)}$ huius fumma ponatur s. Erit $\frac{p\,d(x^\pi s)}{dx} = \frac{p(\pi+1)x^\pi}{\alpha+\beta} + \frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)(2\alpha+\beta)} + \text{------} + \frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\text{-----}(\alpha n+\beta)}$. Fiat $pn + p\pi = \alpha n + \beta$, erit $p = \alpha$, et $\pi = \frac{\alpha}{\beta}$. Quamobrem erit $\frac{\alpha\,d(x^{\frac{\alpha}{\beta}} s)}{dx} = x^{\frac{\alpha}{\beta}} + \frac{x^{\frac{\alpha}{\beta}+1}}{\alpha+\beta}$

$+$ ---

$+ \cdots\cdots + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(n-1)+\beta)}$. Et propterea $\frac{\alpha d(x^{\frac{\beta}{\alpha}} s)}{x^{\frac{\beta}{\alpha}} dx} = 1 + s - \frac{x^n}{A}$ posito vt ante $A = (\alpha+\beta)\cdots\cdots(\alpha n+\beta)$. Aequatio haec euoluta praebit $\alpha x^{\frac{\beta}{\alpha}} ds + \beta x^{\frac{\beta-\alpha}{\alpha}} s\,dx = x^{\frac{\beta}{\alpha}} dx + x^{\frac{\beta}{\alpha}} s\,dx - \frac{x^{\frac{\beta}{\alpha}+n} dx}{A}$ quae diuisa per $x^{\frac{\beta}{\alpha}-1}$ transit in $\alpha x\,ds + \beta s\,dx = x\,dx + x\,s\,dx - \frac{x^{n+1} dx}{A}$, seu $ds + \frac{\beta s\,dx}{\alpha x} - \frac{s\,dx}{\alpha} = \frac{dx}{\alpha} - \frac{x^n\,dx}{A\alpha}$. Multiplicetur haec aequatio per $c^{\frac{-x}{\alpha}} x^{\frac{\beta}{\alpha}}$, vbi c est numerus, cuius log. est 1, fiet ea integrabilis, prodibitque $c^{\frac{-x}{\alpha}} x^{\frac{\beta}{\alpha}} s = \frac{1}{\alpha}\int c^{\frac{-x}{\alpha}} x^{\frac{\beta}{\alpha}} dx(1-\frac{x^n}{A})$. Atque $s = \frac{1}{\alpha} c^{\frac{x}{\alpha}} x^{\frac{-\beta}{\alpha}} \int c^{\frac{-x}{\alpha}} x^{\frac{\beta}{\alpha}} dx (1-\frac{x^n}{A})$. Huius progressionis in infinitum continuatae summa igitur erit $\frac{1}{\alpha} c^{\frac{x}{\alpha}} x^{\frac{-\beta}{\alpha}} \int c^{\frac{-x}{\alpha}} x^{\frac{\beta}{\alpha}} dx = \beta(\beta-\alpha)(\beta-2\alpha)\cdots\cdots \alpha c^{\frac{x}{\alpha}} x^{\frac{-\beta}{\alpha}} - 1 - \frac{\beta}{x} - \frac{\beta(\beta-\alpha)}{x^2} - \cdots\cdots - \frac{\beta(\beta-\alpha)\cdots\alpha}{x^{\frac{\beta}{\alpha}}}$ si fuerit $\beta = 0$, erit summa $= c^{\frac{x}{\alpha}} - 1$. Sin sit $\beta = \alpha$ erit summa $= \frac{\alpha c^{\frac{x}{\alpha}}}{x} - 1 - \frac{\alpha}{x}$. Si vero ponatur $\beta = 2\alpha$, summa seriei erit $\frac{2\alpha^2 c^{\frac{x}{\alpha}}}{x^2} - 1 - \frac{2\alpha}{x} - \frac{2\alpha^2}{x^2}$; et ita porro. Ex quo intelligitur, quoties β sit multiplum ipsius α, summam seriei finita et integrata expressione exhiberi posse.

posse. Si autem $\frac{\beta}{\alpha}$ euadat fractio formula inuenta integrari non potest.

§. 17. Crescat terminus quisque duobu factoribus, habebitur progressio haec $\frac{x}{(\alpha+\beta)} + \frac{x^2}{(\alpha+\beta)\cdots(3\alpha+\beta)} + \frac{x^3}{(\alpha+\beta)\cdots(5\alpha+\beta)} + \cdots + \frac{x^n}{(\alpha+\beta)\cdots(\alpha(2n-1)+\beta)}$ cuius summa ponatur s. Erit $\frac{pd(x^\pi s)}{dx} = \frac{p(\pi+1)x^\pi}{\alpha+\beta} + \frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)\cdots(3\alpha+\beta)} + \cdots \frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\cdots(\alpha(2n-1)+\beta)}$ sit $p\pi+pn = 2\alpha n-\alpha+\beta$, erit $p=2\alpha$ et $\pi = \frac{\beta-\alpha}{2\alpha}$ Idcirco $\frac{2\alpha d(x^{\frac{\beta-\alpha}{2\alpha}}s)}{dx} = x^{\frac{\beta-\alpha}{2\alpha}} + \frac{x^{\frac{\beta+\alpha}{2\alpha}}}{(\alpha+\beta)(2\alpha+\beta)} + \cdots + \frac{x^{\frac{\beta-3\alpha}{2\alpha}+n}}{(\alpha+\beta)\cdots(\alpha(2n-2)+\beta)}$. Atque iterum $\frac{2\alpha p d x^\pi d(x^{\frac{\beta-\alpha}{2\alpha}}s)}{dx^2} = \frac{p(\beta-\alpha+2\alpha\pi)}{2\alpha} x^{\frac{\beta-3\alpha}{2\alpha}+\pi} + \cdots \frac{p(\beta-3\alpha+2\alpha n+2\alpha\pi)x^{\frac{\beta-5\alpha}{2\alpha}+n+\pi}}{2\alpha(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(2n-2)+\beta)}$. Fiat $p\beta - 3p\alpha + 2p\alpha n + 2p\alpha\pi = 4\alpha^2 n - 4\alpha^2 + 2\alpha\beta$, erit $p = 2\alpha$, et $\pi = \frac{1}{2}$. Vnde prodit $\frac{4\alpha^2 d(x^{\frac{1}{2}} d(x^{\frac{\beta-\alpha}{2\alpha}}s))}{dx^2} = \beta x^{\frac{\beta-2\alpha}{2\alpha}} + \cdots + \frac{x^{\frac{\beta-4\alpha}{2\alpha}+n}}{(\alpha+\beta)\cdots(\alpha(2n-3)+\beta)}$

$= \beta x$

$$= \beta x^{\frac{\beta - 2\alpha}{2\alpha}} + x^{\frac{\beta - 2\alpha}{2\alpha}} s - \frac{x^{\frac{\beta - 2\alpha}{2\alpha} + n}}{(\alpha+\beta) \cdots (\alpha(2n-1)+\beta)}$$

Simili modo operatio est instituenda, si terminus quisque pluribus factoribus in denominatore crescat. Nec non satis apparet, si progressio, quam factores denominatorum constituunt, non fuerit arithmetica sed algebraica altioris ordinis, quomodo ad aequationem, ex qua summa determinatur, perueniri oporteat. Scilicet quilibet factor in factores simplices est resoluendus, vt §. 15. factum est, vbi terminus generalis factorum est $(\alpha n+\beta)(\gamma n+\delta)$, qui omnes aequationes ordinis secundi sub se complectitur. At ne hoc quidem opus est si sequenti modo operari libuerit. Vt proposita sit progressio

$$\frac{x}{1} + \frac{x^2}{1.7} + \frac{x^3}{1.7.17.} + \frac{x^4}{1.7.17.31.} + \cdots$$

cuius terminus generalis $\frac{x^n}{1.7.\cdots(2n^2-1)}$, summa huius ponatur s, erit $\frac{pd(x^\pi s)}{dx} = p(\pi+1)x^\pi + \cdots \frac{p(\pi+n)x^{n+\pi-1}}{1.7\cdots(2n^2-1)}$. Atque denuo $\frac{pd(x^\rho d(x^\pi s))}{dx^2} = p(\pi+1)(\pi+\rho)x^{\pi+\rho-1} + \cdots + \frac{p(\pi+n)(n+\pi+\rho-1)x^{n+\pi+\rho-2}}{1.7\cdots(2n^2-1)}$. Fiat $pn^2 + 2p\pi n + p\rho n - pn + p\pi^2 + p\pi\rho - p\pi = 2n^2 - 1$, erit $p = 2$, $4\pi + 2\rho - 2 = 0$ seu $\rho = 1 - 2\pi$. Atque $-2\pi^2 = -1$ seu $\pi = \sqrt{\tfrac{1}{2}}$ et $\rho = 1 - \sqrt{2}$. Quare habebitur $\frac{2d(x^{1-\sqrt{2}}d(x^{\sqrt{\frac{1}{2}}}s))}{dx^2} = x^{-\sqrt{\frac{1}{2}}} + \cdots +$

$\frac{x^{\frac{2n-2-\sqrt{2}}{2}}}{1.7 - - - - (2n^2-4n+1)} = x^{\frac{-\sqrt{2}}{2}} + x^{\frac{-\sqrt{2}}{2}}\left(s - \frac{x^n}{1.7. - - - - (2n^2-1)}\right)$. Summa vero huius feriei in infinitum inuenietur ex hac aequatione $x^{-\sqrt{2}} dx(x^{\sqrt{\frac{1}{2}}}s) + 2x^{1\sqrt{2}} dd(x^{\sqrt{\frac{1}{2}}}s) = (2-2\sqrt{2})x^{\frac{-\sqrt{2}}{2}}dxds + (\sqrt{2}-2)x^{\frac{-2-\sqrt{2}}{2}}sdx^2 + 2x^{\frac{2-\sqrt{2}}{2}}dds + 2\sqrt{2}x^{\frac{-\sqrt{2}}{2}}dsdx + (1-\sqrt{2})x^{\frac{-2-\sqrt{2}}{2}}sdx^2 = x^{\frac{-\sqrt{2}}{2}}dx^2 + x^{\frac{-\sqrt{2}}{2}}sdx^2 = 2x^{\frac{-\sqrt{2}}{2}}dsdx - x^{\frac{-2-\sqrt{2}}{2}}sdx^2 + 2x^{\frac{2-\sqrt{2}}{2}}dds$. Seu $2xdds - \frac{sdx^2}{x} + 2dsdx = dx^2 + sdx^2$, ex qua aequatione irrationalia omnia euanuere.

§. 18. Si factores denominatorum conftituant progreffionem potentiarum, huiusmodi progreffionum fummas inueftigabo: vt fit progreffio propofita $\frac{x}{(\alpha+\beta)^2} + \frac{x^2}{(\alpha+\beta)^2(2\alpha+\beta)^2} + - - - - + \frac{x^n}{(\alpha+\beta)^2 - - - (n\alpha+\beta)^2}$ ponatur fumma s, erit $pd\frac{(x^\pi s)}{dx} = \frac{p(\pi+1)x^\pi}{(\alpha+\beta)^2} + - - - - + \frac{p(\pi+n)x^{n+\pi-1}}{(\alpha+\beta)^2 - - - (n\alpha+\beta)^2}$, fiat $p\pi + pn = \alpha n + \beta$, erit $p = \alpha$ et $\pi = \frac{\beta}{\alpha}$. Propterea $\frac{\alpha d(x^{\frac{\beta}{\alpha}}s)}{\alpha dx} = \frac{x^{\frac{\beta}{\alpha}}}{(\alpha+\beta)} + - - - - - + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2 - - - (n\alpha+\beta)}$. Porro

$\alpha p d$

$$\frac{\alpha p\, d(x^\pi d(x^{\frac{\beta}{\alpha}} s))}{dx^2} = \frac{p(\beta + \alpha\pi)x^{\frac{\beta}{\alpha}+\pi-1}}{\alpha(\alpha+\beta)} + \cdots + \frac{p(\beta + \alpha\pi + \alpha n - \alpha)x^{\frac{\beta}{\alpha}+\pi+n-2}}{\alpha(\alpha+\beta)^2 \cdots (\alpha n + \beta)}.$$ Fiat $p\alpha n + p\beta + p\alpha\pi - p\alpha = \alpha^2 n + \alpha\beta$. Ergo $p = \alpha$, et $\pi = \frac{\alpha}{\alpha} = 1$. Vnde est $$\frac{\alpha^2 d(x d(x^{\frac{\beta}{\alpha}} s))}{dx^2} = x^{\frac{\beta}{\alpha}} + \cdots + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2 \cdots (\alpha(n-1)+\beta)^2} = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}}\left(s - \frac{x^n}{(\alpha+\beta)^2 \cdots (\alpha n+\beta)^2}\right).$$ Et summa progressionis in infinitum determinabitur aequatione $\frac{\alpha^2 d(x d(x^{\frac{\beta}{\alpha}} s))}{x^{\frac{\beta}{\alpha}} dx^2} = 1 + s$. Similiter si factores fuerint cubi summa progressionis $\frac{x}{(\alpha+\beta)^3} + \frac{x^2}{(\alpha+\beta)^3(2\alpha+\beta)^3} +$ etc. in infinitum s inuenietur ex hac aequatione $\frac{\alpha^3 d(x d(x d(x^{\frac{\beta}{\alpha}} s)))}{x^{\frac{\beta}{\alpha}} dx^3} = 1 + s$. Atque ita porro pro sequentibus.

§. 19. Sint nunc coëfficientes potentiarum ipsius x fractiones, quarum tam numeratores quam denominatores sint facta ex certo factorum numero pro indice, cuiusque termini crescente constantia. Ita sit progressio proposita haec $\frac{(a+b)x}{(\alpha+\beta)} + \frac{(a+b)(2a+b)}{(\alpha+\beta)(2\alpha+\beta)}x^2 + \cdots + \frac{(a+b) \cdots (na+b)}{(\alpha+\beta) \cdots (n\alpha+\beta)}x^n$; huius summa ponatur s, erit $p\int x^\pi s\, dx =$

$dx = \frac{p(a+b)}{(\pi+2)(\alpha+\beta)} x^{\pi+2} + \cdots + \frac{p(a+b)\cdots(\alpha n+b)}{(\pi+n+1)(\alpha+\beta)\cdots(\alpha n+\beta)} x^{\pi+n+1}$. Fiat $apn + bp = \pi + n + 1$, erit $p = \frac{1}{a}$ et $\pi = \frac{b-a}{a}$. Adeoque $\frac{\int x^{\frac{b-a}{a}} s\,dx}{a} = \frac{x^{\frac{b+a}{a}}}{\alpha+\beta} + \cdots + \frac{(a+b)\cdots(a(n-1)+b)}{(\alpha+\beta)\cdots(\alpha n+\beta)} x^{\frac{b}{a}+n}$. Et denuo $\frac{pd(x^{\pi}\int x^{\frac{b-a}{a}} s\,dx)}{a\,dx} = \frac{p(b+a+a\pi)}{a(\alpha+\beta)} x^{\frac{b}{a}+\pi} + \cdots + \frac{p(b+an+a\pi)(a+b)\cdots(a(n-1)+b)}{a(\alpha+\beta)\cdots(\alpha n+\beta)} x^{\frac{b}{a}+\pi+n-1}$. Fiat $bp + apn + ap\pi = a\alpha n + a\beta$, erit $p = \alpha$, et $\pi = \frac{\beta}{\alpha} - \frac{b}{a}$. Vnde erit $\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}} s\,dx)}{a\,dx} = x^{\frac{\beta}{\alpha}} + \cdots + \frac{(a+b)\cdots(a(n-1)+b)}{(\alpha+\beta)\cdots(\alpha(n-1)+\beta)} x^{\frac{\beta}{\alpha}+n-1} = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}}\left(s - \frac{(a+b)\cdots(an+b)}{(\alpha+\beta)\cdots(\alpha n+\beta)} x^{n}\right)$. Ex qua aequatione s determinare licet. Si ſumma progreſſionis propoſitae in infinitum deſideretur, erit $\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}} s\,dx)}{a\,dx} = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}} s$, ſeu $\frac{\alpha}{a}\left(\frac{\beta}{\alpha} - \frac{b}{a}\right) x^{\frac{\beta}{\alpha}-\frac{b}{a}-1}\int x^{\frac{b-a}{a}} s\,dx + \frac{\alpha}{a} x^{\frac{\beta}{\alpha}-1} s = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}} s$. Quae abit in hanc $\left(\frac{\beta}{a} - \frac{\alpha b}{aa}\right) x^{-\frac{b}{a}}\int x^{\frac{b-a}{a}} s\,dx + \frac{\alpha}{a} s = x + xs$, vel hanc $\left(\frac{\beta}{a} - \frac{\alpha b}{a^2}\right)\int x^{\frac{b-a}{a}} s\,dx + \frac{\alpha}{a} x^{\frac{b}{a}} s = x^{\frac{b+a}{a}} + x^{\frac{b+a}{a}} s$. Haec differentiata dat $\left(\frac{\beta}{a} - \frac{\alpha b}{a^2}\right) x^{\frac{b-a}{a}} s\,dx + \frac{\alpha}{a} x^{\frac{b}{a}} ds + \frac{\alpha b}{a^2} x^{\frac{b-a}{a}} s\,dx = \left(\frac{b+a}{a}\right) x^{\frac{b}{a}} dx + x^{\frac{b+a}{a}} ds + \left(\frac{b+a}{a}\right) x^{\frac{b}{a}} s\,dx$, quae reducitur ad hanc $\frac{\beta}{a} s\,dx + \frac{\alpha}{a^2}$

$x\,ds$

$x ds = (\frac{b+a}{a}) x dx + x^2 ds + (\frac{b+a}{a}) x s dx$. Seu $ds + \frac{\beta s dx - (b+a) x s dx}{\alpha x - a x^2} = \frac{(b+a) x dx}{\alpha x - a x^2}$. Multiplicetur haec aequatio per $e^{\int \frac{\beta dx - (b+a) x dx}{\alpha x - a x^2}}$, vel per $x^{\frac{\beta}{\alpha}} (\alpha - a x)^{\frac{b}{a} - \frac{\beta}{\alpha} + 1}$. Erit $x^{\frac{\beta}{\alpha}} (\alpha - a x)^{\frac{b}{a} - \frac{\beta}{\alpha} + 1} s = (b+a) \int x^{\frac{\beta}{\alpha}} (\alpha - a x)^{\frac{b}{a} - \frac{\beta}{\alpha}} dx$. Atque $s = \frac{(b+a) \int x^{\frac{\beta}{\alpha}} (\alpha - a x)^{\frac{b}{a} - \frac{\beta}{\alpha}} dx}{x^{\frac{\beta}{\alpha}} (\alpha - a x)^{\frac{b}{a} - \frac{\beta}{\alpha} + 1}}$. Summa igitur algebraice poterit assignari si vel $\frac{\beta}{\alpha}$ vel $\frac{b}{a} - \frac{\beta}{\alpha}$ fuerit numerus integer affirmatiuus.

§. 20. Si progressio fuerit ex huiusmodi ipsius x coefficientibus et algebraicis composita; primo coefficientes algebraici differentiatione et integratione debent tolli, vt ibi est factum, et tum progressio resultans modo hic exposito tractari. Vt sit progressio proposita $\frac{1x}{1} + \frac{3x^2}{1.2} + \frac{5x^3}{1.2.3} + \cdots + \frac{(2n-1)x^n}{1.2.3 \cdots n}$ summa huius ponatur s, erit $p \int x^\pi s dx = \frac{1.p x^{\pi+2}}{(\pi+2)1} + \cdots + \frac{(2n-1) p x^{\pi+n+1}}{(\pi+n+1)(1.2.3 \cdots n)}$. Fiat $2np - p = \pi + n + 1$, erit $p = \frac{1}{2}$ et $\pi = -\frac{3}{2}$. Ex quo erit $\frac{\int x^{-\frac{3}{2}} s dx}{2} = \frac{x^{\frac{1}{2}}}{1} + \frac{x^{\frac{3}{2}}}{1.2} + \cdots \frac{x^{n-\frac{1}{2}}}{1.2.3 \cdots n}$. Multiplicetur per $x^{\frac{1}{2}}$, erit $\frac{x^{\frac{1}{2}} \int x^{-\frac{3}{2}} s dx}{2} = \frac{x}{1} + \frac{x^2}{1.2} +$

$\frac{x^3}{1.2.3}$

$\frac{x^3}{1.2.3} + --- \frac{x^n}{1.2.3---n}$. Ergo $\frac{d(x^{\frac{1}{2}}\int x^{-\frac{3}{2}} s\,dx)}{2\,dx} = 1$
$+ \frac{x}{1} + \frac{x^2}{1.2} + ----+ \frac{x^{n-1}}{1.2.3---(n-1)} = 1 + \frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}} s\,dx}{2}$
$- \frac{x^n}{1.2.3----n}$, ex qua aequatione s inuenietur. Erit autem $\frac{\int x^{-\frac{3}{2}} s\,dx}{4 x^{\frac{1}{2}}} + \frac{s}{2x} = 1 + \frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}} s\,dx}{2} - \frac{x^n}{1.2.3---n}$
Ponatur $1.2.3.----n = A$, erit porro $(1-2x)\int x^{-\frac{3}{2}}$ $s\,dx = 4 x^{-\frac{1}{2}} - \frac{2s}{x^{\frac{1}{2}}} - \frac{4 x^{n+\frac{1}{2}}}{A}$. Summa progreſſionis propoſitae in infinitum continuatae vero definietur ex iſta aequatione $\int x^{-\frac{3}{2}} s\,dx = \frac{4x - 2s}{(1-2x)\sqrt{x}}$; quae differentiata dat $\frac{s\,dx}{x\sqrt{x}} = \frac{2xdx + 4xxdx + sdx - 6sxdx - 2xds + 4x^2ds}{(1-2x)^2 x\sqrt{x}}$, ſeu $x\,dx + 2 x^2\,dx - s\,x\,dx - 2\,s\,x^2\,dx - x\,ds + 2\,x^2\,ds = 0$. Quae reducitur ad hanc $ds + \frac{s\,dx(1+2x)}{1-2x} = \frac{dx(1+2x)}{1-2x}$. Quae multiplicata per $\frac{c^{-x}}{1-2x}$ fit integrabilis, prodit autem $\frac{c^{-x}s}{1-2x} = \frac{\int c^{-x}dx(1+2x)}{(1-2x)^2} = \frac{c^{-x}}{1-2x} - 1$. Atque hinc $s = 1 - c^x(1-2x)$. Quare ſi fuerit $x = \frac{1}{2}$ erit $s = 1$. Adeoque $1 = \frac{1}{1.2} + \frac{3}{1.2.4} + \frac{5}{1.2.3\,8} + \frac{7}{1.2.3.4.16} +$ etc. in infin.

§. 21. Ex his apparet ad quas progreſſiones ſummandas methodus hac diſſertatione expoſita ſe extendat: ſcilicet ad omnes eas progreſſiones, quae comprehen-

duntur

dantur, hoc termino generali $\frac{AP}{BQ}x^{\alpha n+\beta}$, vbi A et B designant terminos ordinis n, quarumcunque progressionum algebraicarum. Et P est factum ex $\gamma n+\delta$ terminis progressionis cuiusque algebraicae, itemque Q est simile factum ex $\varepsilon n+\zeta$ terminis etiam cuiuscunque progressionis algebraicae. Omnino autem summae huiusmodi progressionum tribus modis expositae inuenientur. Vel enim prodit summa prorsus algebraica, vel assignatur quadratura quaepiam, a qua summa pendet. Vel tertio aequatio reperitur, cuius variabiles quantitates s et x penitus non possunt a se inuicem separari, vt saltem constet, vtrum progressio summam habeat algebraicam, an a cuiusdam curuae quadratura pendeat. Quamuis vero haec methodus tam late pateat, tamen innumerae occurrere possunt progressiones per eum non summabiles, quarum quidem vel nullo alio modo summae assignari possunt, vt huius $1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\text{------}+\frac{1}{2^n-1}$, vel quarum summae etiam constant, vt huius $\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+$ etc. termino generali existente $\frac{1}{a^\alpha-1}$, in quo a et α numeros quoscunque integros praeter vnitatem denotant; cuius summam esse $=1$ demonstrauit Celeberrimus *Goldbachius*. Quia autem eius terminus generalis proprie sic dictus non potest exhiberi, mirum non est, eam hac methodo non posse summari.